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The Canonical Perfect Bose Gas in Casimir Boxes

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Abstract

We study the problem of Bose-Einstein condensation in the perfect Bose gas in the canonical ensemble, in anisotropically dilated rectangular parallelpipeds (Casimir boxes). We prove that in the canonical ensemble for these anisotropic boxes there is the same type of generalized Bose-Einstein condensation as in the grand-canonical ensemble for the equivalent geometry. However the amount of condensate in the individual states is different in some cases and so are the fluctuations.

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1 Introduction

Many calculations in the grand-canonical ensemble (GCE) show a dependence of Bose-Einstein condensation (BEC) on the way the infinite volume limit is taken. For example, in [1] and [3] the authors study the the perfect boson gas (PBG) in the GCE in rectangular parallelepipeds whose edges go to infinity at different rates (*Casimir boxes*, see [7]). They showed that this *anisotropic dilation* can modify the standard ground-state BEC, converting it into a *generalized* BEC of type II or III. For a short history of the notion of *generalized* BEC we refer the reader to [1] and [3].

On the other hand, due to the lack of (*strong*) equivalence of ensembles, the PBG in the canonical ensemble (CE) and in the GCE gives different expectations and fluctuations for many observables. For example, it was shown in [5] that for the *isotropic dilation* of the *canonical* PBG the distribution of ground-state occupation number is *different* from the one in the GCE. The same is true for the fluctuations of the occupation numbers, which are shape dependent and are not normal or Gaussian. Therefore this lack of equivalence of ensembles does not allow us to deduce the same shape dependence for BEC in the CE as for its grand-canonical counterpart and so far the question of whether it is true in the CE has not been considered except in a special case [6].

The aim of the present paper is to fill this gap. We study the problem of BEC in the PBG in the CE, in anisotropically dilated rectangular boxes. We shall prove that in the CE for these anisotropic boxes there is the same type of generalized BEC as in the GCE for the equivalent geometry. However the amount of condensate in the individual states is different in some cases and so are the fluctuations.

We would like to note that there is a renewed interest in *generalized* BEC both from the theoretical [14], [12], [18] and experimental [16], [8] point of view. This due to recent experiments which produce "fragmentation" of BEC (see e.g. [9], [13]), that is, the single state condensation can be "smeared out" over two or more quantum states. We return to this point in Section 4.

The structure of the present paper is as follows. In the rest of this section we give the mathematical setting. In Section 2 we collect together the results about PBG in the GCE that we shall need. In Section 3 we study the PBG in the CE for the system of anisotropic parallelpipeds. We start by giving some results which are common to the three cases corresponding to the three characteristic ways of taking the thermodynamic limit. These are determined by how fast the longest edge grows: (a) faster than the square root of the volume, (b) like the square root of the volume and (c) slower than the square root of the volume. In the three subsections of Section 3 we study these cases separately. In Section 4 we discuss the results.

We finish this section by establishing the general setting and notation.

Let Λ_V be a rectangular parallelepiped of volume V :

$$\Lambda_V := \left\{x \in \mathbb{R}^3 : 0 \leq x_j \leq V^{\alpha_j}, j = 1,2,3\right\}, \tag{1.1}$$

where

$$\alpha_1 \geq \alpha_2 \geq \alpha_3 > 0, \text{ and } \alpha_1 + \alpha_2 + \alpha_3 = 1. \tag{1.2}$$

The space of one-particle wave-functions is $\mathcal{H}_V = L^2(\Lambda_V)$ and the one-particle Hamiltonian t_V is the self-adjoint extension of the operator $-\Delta/2$ determined by the Dirichlet boundary conditions on $\partial\Lambda_V$. We denote by $\{E_k(V)\}_{k=1}^{\infty}$ the ordered eigenvalues of t_V:

$$0 < E_1(V) < E_2(V) \le E_3(V) \le \dots \, .$$

We also introduce the boson Fock space on $\mathcal{H}_V$ defined by $\mathcal{F}(\mathcal{H}_V) := \bigoplus_{n=0}^{\infty} \mathcal{H}_{V,\,\mathrm{symm}}^n$, where $\mathcal{H}_{V,\,\mathrm{symm}}^n := \left(\bigotimes_{j=1}^n \mathcal{H}_V\right)_{\mathrm{symm}}$ stands for the space of n-particle symmetric functions. Then $T_V^{(n)}$ denotes the n-particle free Hamiltonian determined by $t_V \equiv T_V^{(1)}$ on $\mathcal{H}_{V,\,\mathrm{symm}}^n$, and T_V the corresponding Hamiltonian in the Fock space.

Now the expectations for the PBG in the canonical ensemble at temperature β^{-1} and density $\rho = n/V$ are defined by the Gibbs state

$$\langle - \rangle_V^{\mathcal{C}}(\rho) := (Z_V(n))^{-1} \,\mathrm{Tr}\,_{\mathcal{H}_{V,\,\mathrm{symm}}^n} (-)\, e^{-\beta T_V^{(n)}}, \tag{1.3}$$

where

$$Z_V(n) := \mathrm{Tr}\,_{\mathcal{H}_{V,\,\mathrm{symm}}^n} e^{-\beta T_V^{(n)}} \tag{1.4}$$

is the n-particle canonical partition function. As usual we put $Z_V(0) = 1$. The grand-canonical Gibbs state is defined by

$$\langle - \rangle_V^{\mathcal{GC}}(\mu) := (\Xi_V(\mu))^{-1} \,\mathrm{Tr}\,_{\mathcal{F}(\mathcal{H}_V)} (-)\, e^{-\beta(T_V - \mu N_V)}, \tag{1.5}$$

where μ is the corresponding chemical potential. Here N_V is the particle number operator, that is, $N_V := \sum_{k\ge 1} N_k$ where N_k denotes the operator for the number of particles in the k-th one-particle state. The grand-canonical partition function at chemical potential $\mu < E_1(V)$ is

$$\Xi_V(\mu) := \mathrm{Tr}\,_{\mathcal{F}(\mathcal{H}_V)} e^{-\beta(T_V - \mu N_V)}. \tag{1.6}$$

Because of their commutative nature it is useful to think of N_V and $\{N_k\}_{k\ge 1}$ as random variables rather than operators.

Notice that the one-particle Hamiltonian spectrum $\sigma(t_V) = \{E_k(V)\}_{k=1}^{\infty}$ coincides with the set

$$\left\{ \epsilon_{\mathbf{n},V} = \frac{\pi^2}{2} \sum_{j=1}^{3} \frac{n_j^2}{V^{2\alpha_j}} : n_j = 1,2,3,\dots ; ,\ j = 1,2,3 \right\}, \tag{1.7}$$

described by the multi-index $\mathbf{n} = (n_1, n_2, n_3)$. Then and the ground-state eigenvalue $E_1(V) = \epsilon_{(1,1,1),V}$.

Let $\{\eta_k(V) := E_k(V) - E_1(V)\}_{k\ge 1}$. For a given V we define $F_V : \mathbb{R} \to \mathbb{R}_+$ by

$$F_V(\eta) := \frac{1}{V} \# \{k : \eta_k(V) \le \eta\} \, . \tag{1.8}$$

Note that F_V is a nondecreasing function on $\mathbb{R}$ with $F_V(\eta) = 0$ for $\eta < 0$. $VF_V(\eta - E_1(V))$ is the distribution of the eigenvalues (*integrated density of states*) of the one-particle Hamiltonian t_V. One can prove in many ways, for example by using Lemma 3.1 or by taking the Laplace transform, that

$$F(\eta) := \lim_{V\to\infty} F_V(\eta) = (\sqrt{2}/3\pi^2)\eta^{3/2} \ , \quad \eta \ge 0 \, . \tag{1.9}$$

We shall show (see Lemma 3.1) that $F_V(\eta) \leq (\sqrt{2}/3\pi^2)\eta^{3/2}$ if $\eta > C/V^{2\alpha_3}$ for some $C > 0$. This bound and (1.9) imply that the *critical* density of the PBG:

$$\rho_c := \lim_{\varepsilon \downarrow 0} \lim_{V \to \infty} \int_{(\varepsilon,\infty)} \frac{1}{e^{\beta\eta} - 1} F_V(d\eta) = \int_0^\infty \frac{1}{e^{\beta\eta} - 1} F(d\eta) < \infty. \tag{1.10}$$

is finite for any non-zero temperature.

By (1.5) the mean occupation number of the PBG in the grand-canonical ensemble in the state k is given by

$$\langle N_k \rangle_V^{\mathcal{GC}}(\mu) = \frac{1}{e^{\beta(E_k(V)-\mu)} - 1}. \tag{1.11}$$

Let $\mu_V(\rho) < E_1(V)$ be the unique root of the equation

$$\rho = \frac{1}{V} \langle N_V \rangle_V^{\mathcal{GC}}(\mu) \tag{1.12}$$

for a given V. Then a standard result [3] shows that the boundedness of the critical density (1.10) implies the existence of *generalized* BEC with condensate density, ρ_0, given by:

$$\rho_0 := \lim_{\varepsilon \downarrow 0} \lim_{V \to \infty} \frac{1}{V} \sum_{\{k : E_k(V) < \varepsilon\}} \langle N_k \rangle_V^{\mathcal{GC}}(\mu_V(\rho)) = \rho - \rho_c, \text{ for } \rho > \rho_c. \tag{1.13}$$

Following the *van den Berg-Lewis-Pulé classification* [1] and [3], it is useful to identify three categories of generalized BEC:
I. The condensation is of *type* I when a *finite* number of single-particle states are macroscopically occupied.
II. It is of *type* II when an *infinite* number of states are macroscopically occupied.
III. It is of *type* III when *none* of the states is macroscopically occupied.
For a specific geometry we have more detailed information at our disposal. In the next section we collect the results from [1] that we shall need later about the GCE in the case of the anisotropically dilated parallelepipeds (1.1).

Remark 1.1 Though we have chosen here to work with Dirichlet boundary conditions, the proofs in this paper can be adapted without difficulty to periodic or Neumann boundary conditions.
Remark 1.2 Note that according to the classification presented above the condensate "fragmentation" is nothing but a generalized BEC of *type* I or II.

2 Generalized Bose-Einstein Condensation of the Perfect Bose Gas in the Grand Canonical Ensemble

Proposition 2.1 ([1],Theorem 1)
Let $\overline{\mu}_V(\rho) = \mu_V(\rho) - E_1(V)$. Then the behaviour of $\overline{\mu}_V(\rho)$ is as follows:

1. *For $\rho \leq \rho_c$, $\lim_{V \to \infty} \overline{\mu}_V(\rho) = \overline{\mu}(\rho)$ where $\overline{\mu}(\rho) < 0$ is the unique root $\overline{\mu}(\rho) < 0$ of the equation*

$$\rho = \lim_{V \to \infty} \frac{1}{V} \langle N_V \rangle_V^{\mathcal{GC}}(\mu) = \int_0^\infty \frac{1}{e^{\beta(\eta-\mu)} - 1} F(d\eta) \,. \tag{2.1}$$

2. *For $\rho > \rho_c$, $\lim_{V\to\infty} \overline{\mu}_V(\rho) = 0$ and for $V \to \infty$,*

$$\overline{\mu}_V(\rho) = -\left\{\beta V(\rho-\rho_c)\right\}^{-1} + \mathrm{O}(1/V), \quad \text{if } \alpha_1 < 1/2; \tag{2.2}$$
$$\overline{\mu}_V(\rho) = -\left\{\beta V A(\rho)\right\}^{-1} + \mathrm{O}(1/V), \quad \text{if } \alpha_1 = 1/2; \tag{2.3}$$
$$\overline{\mu}_V(\rho) = -\left\{2\beta V^{2(1-\alpha_1)}(\rho-\rho_c)^2\right\}^{-1} + \mathrm{O}(1/V), \quad \text{if } \alpha_1 > 1/2, \tag{2.4}$$

where $A(\rho)$ is the unique root of the equation

$$(\rho-\rho_c) = \sum_{j=1}^{\infty}\left[\frac{\pi^2}{2}(j^2-1) + A^{-1}\right]^{-1}. \tag{2.5}$$

The next statement by the same authors shows that there are different *types* of generalized BEC corresponding to different asymptotics (2.2)-(2.4).

Proposition 2.2 ([1])
For $\rho \le \rho_c$ there is no generalized BEC and therefore no BEC of any type.

For $\rho > \rho_c$ there is generalized BEC and all three types of BEC occur:

1. *For $\alpha_1 < 1/2$ only the ground-state is macroscopically occupied (BEC of type I):*

$$\lim_{V\to\infty}\frac{1}{V}\langle N_{\mathbf{n}}\rangle_V^{\mathcal{GC}}(\mu_V(\rho)) = \begin{cases} \rho-\rho_c, & \textit{for } \mathbf{n} = (1,1,1), \\ 0, & \textit{for } \mathbf{n} \neq (1,1,1). \end{cases} \tag{2.6}$$

2. *For $\alpha_1 = 1/2$ there is macroscopic occupation of an infinite number of low-lying levels (BEC of type II):*

$$\lim_{V\to\infty}\frac{1}{V}\langle N_{\mathbf{n}}\rangle_V^{\mathcal{GC}}(\mu_V(\rho)) = \begin{cases} \left\{(n_1^2-1)\pi^2/2 + A^{-1}\right\}^{-1}, & \textit{for } \mathbf{n} = (n_1,1,1), \\ 0, & \textit{for } \mathbf{n} \neq (n_1,1,1). \end{cases} \tag{2.7}$$

3. *Finally, for $\alpha_1 > 1/2$ no single-particle state is macroscopically occupied (BEC of type III):*

$$\lim_{V\to\infty} V^{-1}\langle N_{\mathbf{n}}\rangle_V^{\mathcal{GC}}(\mu_V(\rho)) = 0, \tag{2.8}$$
$$\lim_{V\to\infty} V^{2(\alpha_1-1)}\langle N_{\mathbf{n}}\rangle_V^{\mathcal{GC}}(\mu_V(\rho)) = \begin{cases} 2(\rho-\rho_c)^2, & \textit{for } \mathbf{n} = (n_1,1,1), \\ 0, & \textit{for } \mathbf{n} \neq (n_1,1,1). \end{cases} \tag{2.9}$$

We shall need an easy generalization of the foregoing proposition to obtain the distribution of the random variables N_k through their Laplace transform.

Theorem 2.1 *Let $\rho > \rho_c$. Then:*

1. *For $\alpha_1 < 1/2$,*

$$\lim_{V\to\infty}\left\langle \exp\left(-\lambda\frac{N_{\mathbf{n}}}{V}\right)\right\rangle_V^{\mathcal{GC}}(\mu_V(\rho)) = \begin{cases} \dfrac{1}{1+\lambda(\rho-\rho_c)}, & \textit{for } \mathbf{n} = (1,1,1), \\ 0, & \textit{for } \mathbf{n} \neq (1,1,1). \end{cases} \tag{2.10}$$

2. For $\alpha_1 = 1/2$,

$$\lim_{V\to\infty} \left\langle \exp\left(-\lambda \frac{N_{\mathbf{n}}}{V}\right)\right\rangle_V^{\mathcal{GC}} (\mu_V(\rho)) = \begin{cases} \dfrac{(n_1^2-1)\pi^2/2 + A^{-1}}{(n_1^2-1)\pi^2/2 + A^{-1}+\lambda}, & \textit{for } \mathbf{n} = (n_1,1,1), \\ 0, & \textit{for } \mathbf{n} \neq (n_1,1,1). \end{cases} \tag{2.11}$$

3. For $\alpha_1 > 1/2$,

$$\lim_{V\to\infty} \left\langle \exp\left(-\lambda \frac{N_{\mathbf{n}}}{V}\right)\right\rangle_V^{\mathcal{GC}} (\mu_V(\rho)) = 1, \tag{2.12}$$

$$\lim_{V\to\infty} \left\langle \exp\left(-\lambda \frac{N_{\mathbf{n}}}{V^{2(1-\alpha_1)}}\right)\right\rangle_V^{\mathcal{GC}} (\mu_V(\rho)) = \begin{cases} \dfrac{1}{1+2\lambda\left(\rho-\rho_c\right)^2}, & \textit{for } \mathbf{n} = (n_1,1,1), \\ 0, & \textit{for } \mathbf{n} \neq (n_1,1,1). \end{cases} \tag{2.13}$$

Proof: This follows easily from Proposition 2.1 and the identity:

$$\langle \exp\left(-\lambda N_k\right)\rangle_V^{\mathcal{GC}} (\mu_V(\rho)) = \frac{1-e^{-\beta(\eta_k(V)-\overline{\mu}_V(\rho))}}{1-e^{-\beta(\eta_k(V)-\overline{\mu}_V(\rho)+\lambda/\beta)}} \tag{2.14}$$

□

We shall require some properties of the *Kac distribution* $\mathbb{K}_\Lambda(\mu; d\rho)$, see e.g. [1, 3, 10, 19]. The Kac distribution relates the canonical (1.3) and grand-canonical (1.5) expectations in a finite volume:

$$\langle - \rangle_V^{\mathcal{GC}} (\mu) = \int\limits_{[0,\infty)} \langle - \rangle_V^{\mathcal{C}} (x) \mathbb{K}_V(\mu; dx). \tag{2.15}$$

The *limiting* Kac distribution gives the decomposition of the limiting grand-canonical state $\langle - \rangle^{\mathcal{GC}} (\mu)$ into limiting canonical states $\langle - \rangle^{\mathcal{C}} (\rho)$. In the particular case of the PBG it is more convenient to define the Kac distribution in terms of the mean particle density, rather then the chemical potential. Therefore we define

$$\tilde{\mathbb{K}}_V(\rho\,; dx) := \mathbb{K}_V(\mu_V(\rho); dx) \tag{2.16}$$

so that

$$\langle - \rangle_V^{\mathcal{GC}} (\mu_V(\rho)) = \int\limits_{[0,\infty)} \langle - \rangle_V^{\mathcal{C}} (x) \tilde{\mathbb{K}}_V(\rho\,; dx). \tag{2.17}$$

The next proposition proved in [1] gives the limiting Kac density for anisotropically dilated parallelepipeds:

Proposition 2.3 *Let*

$$\tilde{\mathbb{K}}(\rho\,; dx) := \lim_{V\to\infty} \tilde{\mathbb{K}}_V(\rho\,; dx). \tag{2.18}$$

If $\rho \leq \rho_c$, *then the PBG limiting Kac distribution has the one-point support:*

$$\tilde{\mathbb{K}}(\rho\,; dx) = \delta_\rho(dx). \tag{2.19}$$

If $\rho > \rho_c$, *then:*

1. *For* $\alpha_1 < 1/2$,

$$\tilde{\mathbb{K}}(\rho\,; dx) = \begin{cases} 0, & \text{for } x < \rho_c, \\ \dfrac{1}{\rho - \rho_c} \exp\left(-\dfrac{x - \rho_c}{\rho - \rho_c}\right) dx, & \text{for } x > \rho_c. \end{cases} \tag{2.20}$$

2. *For* $\alpha_1 = 1/2$,

$$\tilde{\mathbb{K}}(\rho\,; dx) = \begin{cases} 0, & \text{for } x < \rho_c, \\ \dfrac{\pi^2 \sinh(2/A - \pi)^{\frac{1}{2}}}{(2/A - \pi)^{\frac{1}{2}}} & \\ \times \displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} n^2 \exp\left(-(x - \rho_c)\frac{\pi^2}{2}\left(n^2 + \frac{2}{A\pi^2} - 1\right)\right) dx, & \text{for } x > \rho_c. \end{cases} \tag{2.21}$$

3. *For* $\alpha_1 > 1/2$,

$$\tilde{\mathbb{K}}(\rho\,; dx) = \delta_\rho(dx). \tag{2.22}$$

3 Generalized Bose-Einstein Condensation and Fluctuations of the Perfect Bose Gas in the Canonical Ensemble

In this section we prove results for the CE analogous to those for the GCE. We are forced to use different methods for the three regimes, so we treat them in separate subsections. But first we give some results which will be useful in all three cases. The basic identity for the canonical expectations at density $\rho = n/V$ of the occupation numbers is (see [5] equation (10)):

$$\langle \exp(-\lambda N_k) \rangle_V^{c}(\rho) = e^{\lambda} - (e^{\lambda} - 1) Z_V(n)^{-1} \sum_{m=0}^{n} e^{-(\beta E_k + \lambda)(n-m)} Z_V(m). \tag{3.1}$$

The canonical expectations are notoriously difficult to calculate and are only accessible through the grand-canonical expectations. In the two cases $\alpha_1 < 1$ and $\alpha_1 = 1$ we shall exploit the fact that the sum on the righthand side of equation (3.1) is very similar to the grand-canonical partition function.
The next theorem shows that the canonical expectations are monotonic increasing in the density. Note that this theorem holds for the PBG with any one-particle spectrum.

Theorem 3.1 *For fixed* $k \geq 1$ *and fixed* V, *the canonical expectations for the PBG,* $\langle \exp(-\lambda N_k) \rangle_V^{c}(\rho)$, *are monotonic decreasing functions of the density* ρ *for* $\lambda > 0$ *while the moments* $\langle N_k^r \rangle_V^{c}(\rho)$, $r \geq 1$ *are monotonic increasing functions of the density.*

Proof: From (3.1) we get

$$\begin{aligned} &\langle \exp(-\lambda N_k) \rangle_V^{c}((n+1)/V) - \langle \exp(-\lambda N_k) \rangle_V^{c}(n/V) \\ &= -(e^{\lambda} - 1) \Bigg\{ e^{-(\beta E_k + \lambda)(n+1)} Z_V(n+1)^{-1} \\ &\qquad\qquad + \sum_{m=0}^{n} e^{-(\beta E_k + \lambda)(n-m)} \left(\frac{Z_V(m+1)}{Z_V(n+1)} - \frac{Z_V(m)}{Z_V(n)} \right) \Bigg\}. \end{aligned} \tag{3.2}$$

Since

$$\left(\frac{Z_V(m+1)}{Z_V(n+1)} - \frac{Z_V(m)}{Z_V(n)}\right) = \frac{Z_V(m)}{Z_V(n+1)}\left(\frac{Z_V(m+1)}{Z_V(m)} - \frac{Z_V(n+1)}{Z_V(N)}\right),$$

by the inequalities (see [10]):

$$\frac{Z_V(m+1)}{Z_V(m)} \geq \frac{Z_V(m+2)}{Z_V(m+1)} \geq \ldots \geq \frac{Z_V(n+1)}{Z_V(n)},$$

and by (3.2) we get the monotonicity:

$$\langle \exp(-\lambda N_k)\rangle_V^{\mathcal{C}}((n+1)/V) - \langle \exp(-\lambda N_k)\rangle_V^{\mathcal{C}}(n/V) \leq 0. \tag{3.3}$$

By differentiating (3.2) r times with respect to λ at $\lambda = 0$,

$$\begin{aligned} &\langle N_k^r\rangle_V^{\mathcal{C}}((n+1)/V) - \langle N_k^r\rangle_V^{\mathcal{C}}(n/V) \\ &= \{n^r - (n-1)^r\}e^{-\beta E_k(n+1)}Z_V(n+1)^{-1} \\ &\qquad + \sum_{m=0}^{n}\{(n-m)^r - (n-m-1)^r\}e^{-\beta E_k(n-m)}\left(\frac{Z_V(m+1)}{Z_V(n+1)} - \frac{Z_V(m)}{Z_V(n)}\right). \end{aligned} \tag{3.4}$$

which is positive by the same argument. □

Remark: In this paper whenever we take the limit

$$\lim_{V\to\infty} \langle - \rangle_V^{\mathcal{C}}(\rho) \tag{3.5}$$

we shall mean that we take the system with n particles in a container of volume $V_n = n\rho$ and then let $n \to \infty$, that is,

$$\lim_{n\to\infty} \langle - \rangle_{n\rho}^{\mathcal{C}}(\rho). \tag{3.6}$$

The next theorem is valid for containers of any geometry and not just for rectangular boxes.

Theorem 3.2 *For $\rho \geq \rho_c$ the generalized condensate in the CE at density ρ is equal to $\rho - \rho_c$, that is*

$$\lim_{\varepsilon\downarrow 0}\lim_{V\to\infty}\sum_{\eta_k<\varepsilon}\langle N_k/V\rangle_V^{\mathcal{C}}(\rho) = \rho - \rho_c. \tag{3.7}$$

Proof: The statement is true for the imperfect (mean-field) Bose gas in the GCE, see [2]. Since the mean-field term in the CE is irrelevant, the theorem follows from monotonicity and the fact that the Kac density for the imperfect Bose gas has one-point support. □

In the next theorem we shall make certain assumptions that are clearly satisfied for the parallelpipeds we are considering. We believe that in fact they hold much more generally.

Theorem 3.3 *Suppose that $\lim_{V\to\infty}\langle N_k/V\rangle_V^{\mathcal{GC}}(\mu_\Lambda(\rho'))$ and $\tilde{\mathbb{K}}(\rho';[\rho,\infty))$ are continuous in ρ' at ρ and that $\tilde{\mathbb{K}}(\rho;[\rho,\infty)) \neq 0$. Then $\lim_{V\to\infty}\langle N_k/V\rangle_V^{\mathcal{GC}}(\mu_V(\rho)) = 0$ implies that $\lim_{V\to\infty}\langle N_k/V\rangle_V^{\mathcal{C}}(\rho) = 0$.*

Proof: Using the decomposition (2.17) and monotonicity we get for any $\varepsilon > 0$:

$$\begin{aligned}
\lim_{V\to\infty} \langle N_k/V\rangle_V^{\mathcal{GC}}(\mu_\Lambda(\rho+\varepsilon)) &= \lim_{V\to\infty} \int\limits_{[0,\infty)} \langle\ N_k/V\rangle_V^{\mathcal{C}}(x)\, \tilde{\mathbb{K}}_V(\rho+\varepsilon; dx) \\
&\geq \lim_{V\to\infty} \int\limits_{[\rho,\infty)} \langle N_k/V\rangle_V^{\mathcal{C}}(x)\, \tilde{\mathbb{K}}_V(\rho+\varepsilon; dx) \\
&\geq \limsup_{V\to\infty} \left\{ \langle N_k/V\rangle_V^{\mathcal{C}}(\rho)\, \tilde{\mathbb{K}}_V(\rho+\varepsilon; [\rho,\infty)) \right\} \\
&= \limsup_{V\to\infty} \langle N_k/V\rangle_V^{\mathcal{C}}(\rho)\, \tilde{\mathbb{K}}(\rho+\varepsilon; [\rho,\infty)).
\end{aligned}$$

Since $\lim_{V\to\infty} \langle N_k/V\rangle_V^{\mathcal{GC}}(\mu_\Lambda(\rho'))$ and $\tilde{\mathbb{K}}(\rho'; [\rho,\infty))$ are continuous in ρ', letting ε tend to zero, we get

$$\lim_{V\to\infty} \langle N_k/V\rangle_V^{\mathcal{GC}}(\mu_\Lambda(\rho)) \geq \limsup_{V\to\infty} \langle N_k/V\rangle_V^{\mathcal{C}}(\rho)\, \tilde{\mathbb{K}}(\rho; [\rho,\infty))$$

and because $\tilde{\mathbb{K}}(\rho; [\rho,\infty))$ does not vanish the result follows. $\square$

Remark: Note that this lemma implies that for $\rho \leq \rho_c$, there is never BEC in the CE.

Before looking at the three cases $\alpha_1 < 1/2$, $\alpha_1 = 1/2$ and $\alpha_1 > 1/2$ we first obtain lower and upper bounds on the density of states.

Lemma 3.1

$$\frac{\sqrt{2}}{3\pi^2}\left(\eta^{1/2} - CV^{-\alpha_3}\right)^3 < F_V(\eta) < \frac{\sqrt{2}}{3\pi^2}(\eta + E_1(V))^{3/2} \tag{3.8}$$

for some C.

Proof:

$$VF_V(\eta - E_1(V)) = \#\left\{\mathbf{n} \mid \mathbf{n} \in \mathbb{N}^3, \frac{\pi^2}{2}\sum_{j=1}^{d=3} \frac{n_j^2}{V^{2\alpha_j}} < \eta\right\}, \tag{3.9}$$

that is $VF_V(\eta - E_1(V))$ is the number of points of $\mathbb{N}^3$ inside the ellipsoid

$$\frac{x^2}{2V^{2\alpha_1}\eta/\pi^2} + \frac{y^2}{2V^{2\alpha_2}\eta/\pi^2} + \frac{z^2}{2V^{2\alpha_3}\eta/\pi^2} = 1. \tag{3.10}$$

If we associate the point $\mathbf{n}$ with the volume of the unit cube centered at $\mathbf{n} - (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ we see that this number is less the volume of the ellipsoid in the first octant which is equal to $\pi(2V^{2\alpha_1}\eta/\pi^2)^{\frac{1}{2}}(2V^{2\alpha_2}\eta/\pi^2)^{\frac{1}{2}}(2V^{2\alpha_3}\eta/\pi^2)^{\frac{1}{2}}/6 = \sqrt{2}\eta^{3/2}V/3\pi^2$. Thus

$$VF_V(\eta - E_1(V)) < \frac{\sqrt{2}}{3\pi^2}\eta^{3/2}V \tag{3.11}$$

and so

$$F_V(\eta) < \frac{\sqrt{2}}{3\pi^2}(\eta + E_1(V))^{3/2}. \tag{3.12}$$

Let $a > b > c > 0$, let $\lambda = 1 - 3/c$ and $a' = \lambda a$, $b' = \lambda b$, and $c' = \lambda c$. If the point in the first quadrant (x, y, z) satisfies $x^2/a'^2 + y^2/b'^2 + z^2/c'^2 \leq 1$, then it satisfies $(x+1)^2/a^2 + (y+1)^2/b^2 + (z+1)^2/c^2 \leq 1$. That is, each point inside the first quadrant of the ellipsoid $x^2/a'^2 + y^2/b'^2 + z^2/c'^2 = 1$ lies in a unit cube with the corner $\mathbf{n} \in \mathbb{N}^3$ (with $n_1 > x$, $n_2 > y$ and $n_3 > y$) inside the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. Therefore

$$VF_V(\eta - E_1(V)) > \frac{\sqrt{2}}{3\pi^2} V \left(\eta^{1/2} - \frac{3\pi}{\sqrt{2} V^{\alpha_3}} \right)^3 \tag{3.13}$$

yielding

$$F_V(\eta) > \frac{\sqrt{2}}{3\pi^2} \left((\eta + E_1(V))^{1/2} - \frac{3\pi}{\sqrt{2} V^{\alpha_3}} \right)^3 > \frac{\sqrt{2}}{3\pi^2} V \left(\eta^{1/2} - \frac{3\pi}{\sqrt{2} V^{\alpha_3}} \right)^3. \tag{3.14}$$

□

3.1 Case $\alpha_1 > 1/2$.

We study this case first because it is the simplest since the limiting Kac distribution is a delta measure concentrated at ρ and we have strong equivalence of ensembles (see Proposition 2.3). We shall use this fact together with the monotonicity properties of Theorem 3.1 to show that in this case the limiting canonical and grand-canonical expectations are identical.

Lemma 3.2 *For $\alpha_1 > 1/2$ and $\lambda > 0$ the following inequalities hold*

$$\begin{aligned}
&\liminf_{\varepsilon \downarrow 0} \lim_{V \to \infty} \left\langle \exp\left(-\lambda \frac{N_k}{V^{2(1-\alpha_1)}} \right) \right\rangle_V^{\mathcal{GC}} (\mu_V(\rho - \varepsilon)) \\
&\geq \limsup_{V \to \infty} \left\langle \exp\left(-\lambda \frac{N_k}{V^{2(1-\alpha_1)}} \right) \right\rangle_V^{\mathcal{C}} (\rho) \\
&\geq \liminf_{V \to \infty} \left\langle \exp\left(-\lambda \frac{N_k}{V^{2(1-\alpha_1)}} \right) \right\rangle_V^{\mathcal{C}} (\rho) \\
&\geq \limsup_{\epsilon \downarrow 0} \lim_{V \to \infty} \left\langle \exp\left(-\lambda \frac{N_k}{V^{2(1-\alpha_1)}} \right) \right\rangle_V^{\mathcal{GC}} (\mu_\Lambda(\rho + \epsilon)).
\end{aligned} \tag{3.15}$$

Proof: We start with the first inequality. Using the decomposition (2.17) we get for any $\varepsilon > 0$:

$$
\begin{aligned}
&\lim_{V\to\infty}\left\langle \exp\left(-\lambda\frac{N_k}{V^{2(1-\alpha_1)}}\right)\right\rangle_V^{gc}(\mu_V(\rho-\varepsilon))\\
&=\lim_{V\to\infty}\int_{[0,\infty)}\left\langle \exp\left(-\lambda\frac{N_k}{V^{2(1-\alpha_1)}}\right)\right\rangle_V^{c}(x)\,\tilde{\mathbb{K}}_V(\rho-\varepsilon;dx)\\
&\geq \lim_{V\to\infty}\int_{[0,\rho)}\left\langle \exp\left(-\lambda\frac{N_k}{V^{2(1-\alpha_1)}}\right)\right\rangle_V^{c}(x)\,\tilde{\mathbb{K}}_V(\rho-\varepsilon;dx)\\
&\geq \limsup_{V\to\infty}\left\{\left\langle \exp\left(-\lambda\frac{N_k}{V^{2(1-\alpha_1)}}\right)\right\rangle_V^{c}(\rho)\,\tilde{\mathbb{K}}_V(\rho-\varepsilon;[0,\rho)\right\}\\
&=\limsup_{V\to\infty}\left\langle \exp\left(-\lambda\frac{N_k}{V^{2(1-\alpha_1)}}\right)\right\rangle_V^{c}(\rho).
\end{aligned}
\tag{3.16}
$$

In the penultimate inequality equality we have used the monotonicity established in Theorem 3.1 and in the last one we have used (2.19) and (2.22). The last inequality in (3.15) is proved similarly:

$$
\begin{aligned}
&\left\langle \exp\left(-\lambda\frac{N_k}{V^{2(1-\alpha_1)}}\right)\right\rangle_V^{gc}(\mu_V(\rho+\varepsilon))\\
&=\int_{[0,\infty)}\left\langle \exp\left(-\lambda\frac{N_k}{V^{2(1-\alpha_1)}}\right)\right\rangle_V^{c}(x)\,\tilde{\mathbb{K}}_V(\rho+\varepsilon;dx)\\
&=\int_{[0,\rho)}\left\langle \exp\left(-\lambda\frac{N_k}{V^{2(1-\alpha_1)}}\right)\right\rangle_V^{c}(x)\,\tilde{\mathbb{K}}_V(\rho+\varepsilon;dx)\\
&\qquad\qquad+\int_{[\rho,\infty)}\left\langle \exp\left(-\lambda\frac{N_k}{V^{2(1-\alpha_1)}}\right)\right\rangle_V^{c}(x)\,\tilde{\mathbb{K}}_V(\rho+\varepsilon;dx)\\
&\leq \tilde{\mathbb{K}}_V(\rho+\varepsilon;[0,\ \rho))+\left\langle \exp\left(-\lambda\frac{N_k}{V^{2(1-\alpha_1)}}\right)\right\rangle_V^{c}(\rho)\tilde{\mathbb{K}}_V(\rho+\varepsilon;[\rho,\ \infty)).
\end{aligned}
\tag{3.17}
$$

Therefore

$$
\begin{aligned}
&\lim_{V\to\infty}\left\langle \exp\left(-\lambda\frac{N_k}{V^{2(1-\alpha_1)}}\right)\right\rangle_V^{gc}(\mu_V(\rho+\varepsilon))\\
&\leq \liminf_{V\to\infty}\left\langle \exp\left(-\lambda\frac{N_k}{V^{2(1-\alpha_1)}}\right)\right\rangle_V^{c}(\rho).
\end{aligned}
\tag{3.18}
$$

$\square$

The following theorem and corollary give the distribution and the mean of $N_{\mathbf{n}}/V^{2(1-\alpha_1)}$ and therefore they give the fluctuations about the mean.

Theorem 3.4 *If $\alpha_1 > 1/2$ and $\rho > \rho_c$ then the limiting distribution in the canonical ensemble of $N_{\mathbf{n}}/V^{2(1-\alpha_1)}$ has Laplace transform for $\lambda > 0$ given by*

$$\lim_{V\to\infty}\left\langle \exp\left(-\lambda\frac{N_{\mathbf{n}}}{V^{2(1-\alpha_1)}}\right)\right\rangle_V^{c}(\rho) = \begin{cases} \dfrac{1}{1+2\lambda\left(\rho-\rho_c\right)^2}, & \textit{for } \mathbf{n}=(n_1,1,1), \\ 0, & \textit{for } \mathbf{n}\neq(n_1,1,1). \end{cases} \tag{3.19}$$

Proof: This follows from the preceding lemma and Theorem 2.1. □

Corollary 3.1 *If $\alpha_1 > 1/2$ and $\rho > \rho_c$ then*

$$\lim_{V\to\infty}\left\langle \frac{N_{\mathbf{n}}}{V^{2(1-\alpha_1)}}\right\rangle_V^{c}(\rho) = \begin{cases} 2\lambda\left(\rho-\rho_c\right)^2, & \textit{for } \mathbf{n}=(n_1,1,1), \\ 0, & \textit{for } \mathbf{n}\neq(n_1,1,1). \end{cases} \tag{3.20}$$

Proof: Again we have to check that there exists $K < \infty$ such that for all V

$$\left\langle \left(\frac{N_k}{V^{2(1-\alpha_1)}}\right)^2\right\rangle_V^{c}(\rho) < K. \tag{3.21}$$

Then the corollary follows from the preceding theorem. We have again for any $\varepsilon > 0$:

$$\begin{aligned} &\left\langle \left(\frac{N_k}{V^{2(1-\alpha_1)}}\right)^2\right\rangle_V^{\mathcal{GC}}(\mu_V(\rho+\epsilon)) \\ &= \int_{[0,\infty)} \left\langle \left(\frac{N_k}{V^{2(1-\alpha_1)}}\right)^2\right\rangle_V^{c}(x)\,\tilde{\mathbb{K}}_V(\rho+\epsilon;dx) \\ &\geq \int_{[\rho,\infty)} \left\langle \left(\frac{N_k}{V^{2(1-\alpha_1)}}\right)^2\right\rangle_V^{c}(x)\,\tilde{\mathbb{K}}_V(\rho+\epsilon;dx) \\ &\geq \left\langle \left(\frac{N_k}{V^{2(1-\alpha_1)}}\right)^2\right\rangle_V^{c}(\rho)\,\tilde{\mathbb{K}}_V(\rho+\epsilon;[\rho,\infty)) \\ &\geq \frac{1}{2}\left\langle \left(\frac{N_k}{V^{2(1-\alpha_1)}}\right)^2\right\rangle_V^{c}(\rho), \end{aligned} \tag{3.22}$$

if V is large enough. This implies the existence of K as above since

$$\left\langle \left(\frac{N_k}{V^{2(1-\alpha_1)}}\right)^2\right\rangle_V^{\mathcal{GC}}(\mu_V(\rho+\epsilon)) \tag{3.23}$$

converges as $V \to \infty$. □

Corollary 3.2 *When $\alpha_1 > 1/2$, there is type III BEC in the canonical ensemble.*

Proof: From the preceding corollary or from Theorem 3.3 we can deduce immediately that

$$\lim_{V\to\infty} \left\langle \frac{N_k}{V} \right\rangle_V^c (\rho) = 0, \tag{3.24}$$

for any $\rho > \rho_c$. $\square$

3.2 Case $\alpha_1 = 1/2$.

For this case BEC into the ground state is treated in [6]. Here we extend the result to higher levels.

Because for $\alpha_1 = 1/2$ the spectral series (1.7) corresponding to $n_1 = 1, 2, 3, ...$ has the *smallest* energy spacing $\pi^2/2V$, it plays a specific role in calculations of the limiting occupation densities. Let

$$\epsilon_n := \pi^2 n^2/2\,,\ \ \eta_{m,n} := \beta(\epsilon_m - \epsilon_n)\,,\ \ b_{m,n} := \eta_{m,n}^{-1} \prod_{\{m'\neq n,\, m'\neq m\}} (1 - \eta_{m,n}/\eta_{m',n})^{-1} \tag{3.25}$$

for $m \neq n$.
In ([6]) the following result was proved:

Let $\alpha_1 = 1/2$. Then for $\rho > \rho_c$

$$\lim_{V\to\infty} \left\langle \frac{N_1}{V} \right\rangle_V^c (\rho) = \frac{\displaystyle\sum_{m=2}^{\infty} b_{m,1} \left\{\eta_{m,1}(\rho-\rho_c) - 1 + \exp\left[-\eta_{m,1}(\rho-\rho_c)\right]\right\}}{\displaystyle\sum_{m=2}^{\infty} b_{m,1}\eta_{m,1} \left\{1 - \exp\left[-\eta_{m,1}(\rho-\rho_c)\right]\right\}}, \tag{3.26}$$

Here we give an extension of (3.26) to other k's. Note that by Theorem 3.3 and comparison with the GCE, in this case there can only be condensation in states corresponding to $\mathbf{n} = (n_1, 1, 1)$. The main tool in the technique developed in [5] and [6] is the following identity:

Let $\{K_{k,V}(dx)\}_{k\geq 1}$ be (non-normalized) measures whose distributions are the functions

$$K_{k,V}(x) = \begin{cases} Z_V(r)\exp\left\{-\beta(Vp_k - rE_k(V)\right\}, & \text{for } r/V < x \leq (r+1)/V, \\ 0, & \text{for } x \leq 0, \end{cases} \tag{3.27}$$

for $r = 0, 1, 2, \ldots$ and some $\{p_k\}_{k\geq 1}$. Then we can re-write equation (3.1) as follows

$$\begin{aligned} \langle \exp\{-\lambda N_k/V\}\rangle_V^c(\rho) &= e^{-\lambda\rho} \int_{[0,\,\rho+1/V]} e^{\lambda x} K_{k,V}(dx)/K_{k,V}(\rho+1/V) \\ &= e^{-\lambda/V} - \lambda e^{-\lambda\rho} \int_0^{\rho+1/V} K_{k,V}(x)e^{\lambda x}dx/K_{k,V}(\rho+1/V). \end{aligned} \tag{3.28}$$

We shall use this identity to calculate the thermodynamic limit of its left-hand side for a given density and $k \geq 1$. From (3.27) we can calculate the Laplace transformation of the measure $K_{k,V}(dx)$:

$$\int_{\mathbb{R}} e^{-\lambda x} K_{k,V}(dx) = (1 - e^{-\lambda/V}) e^{-V\beta p_k}\, \Xi_V(E_k(V) - \lambda/\beta V) \tag{3.29}$$

where for the PBG the grand-canonical partition function (1.6) has the explicit form:

$$\Xi_V(\mu) = \prod_{k=1}^{\infty} \left\{1 - e^{-\beta(E_k(V)-\mu)}\right\}^{-1}.$$

Now we fix the p_k's which are still arbitrary by the defining

$$p_k := -\frac{1}{\beta V} \sum_{j \neq k} \ln \left|1 - e^{-\beta(E_j(V)-E_k(V))}\right| \tag{3.30}$$

and prove the following lemma. Let

$$\tilde{K}_{n,V} := K_{(n,1,1),V}. \tag{3.31}$$

Lemma 3.3 *Let $\alpha_1 = 1/2$. Then*

$$\begin{aligned} \tilde{K}_n(x) &:= \lim_{V\to\infty} \tilde{K}_{n,V}(x) \qquad (3.32)\\ &= \begin{cases} 0 & \textit{for } x \leq \rho_c, \\ (-1)^{(n-1)} \displaystyle\sum_{m=1,\, m\neq n}^{\infty} b_{m,n}\eta_{m,n} \left\{1 - \exp\left[-\eta_{m,n}(x-\rho_c)\right]\right\} & \textit{for } x > \rho_c. \end{cases} \end{aligned}$$

Proof: Let $\lambda > V(E_k - E_1)$. From the definitions (3.29) and (3.30) we get:

$$\int_{\mathbb{R}} e^{-\lambda x} K_{k,V}(dx) = \prod_{j\neq k} \frac{\left|1 - e^{-\beta(E_j(V)-E_k(V))}\right|}{1 - e^{-\beta(E_j(V)-E_k(V)+\lambda/\beta V)}} = \exp\left\{-\sum_{j\neq k} \ln\left|1 + \frac{1-e^{-\lambda/V}}{e^{\beta(E_j(V)-E_k(V))}-1}\right|\right\}.$$

For $k \geq 1$ and $\eta \geq \eta_1(V) - \eta_k(V)$ we define the *shifted* integrated density of states, cf (1.8)

$$F_{k,V}(\eta \geq 0) := \frac{1}{V}\#\left\{j : \eta_j(V) \leq \eta + \eta_k(V),\ j \neq k\right\} = F_V(\eta + \eta_k(V)) - \frac{1}{V}\mathbf{1}_{[0,\infty)}(\eta), \tag{3.33}$$

For $b > a$, let

$$I_{k,V}(a,b) := V \int_{[a,b)} \ln\left|1 + \frac{1-e^{-\lambda/V}}{e^{\beta\eta}-1}\right| F_{k,V}(d\eta)\,. \tag{3.34}$$

Then

$$\begin{aligned} \sum_{j\neq k} \ln\left|1 + \frac{1-e^{-\lambda/V}}{e^{\beta(E_j(V)-E_k(V))}-1}\right| &= V \int_{[\eta_1(V)-\eta_k(V),\,\infty)} \ln\left|1 + \frac{1-e^{-\lambda/V}}{e^{\beta\eta}-1}\right| F_{k,V}(d\eta) \\ &= I_{k,V}(\eta_1(V)-\eta_k(V), \infty)\,. \qquad (3.35) \end{aligned}$$

We can write

$$I_{k,V}(\eta_1(V)-\eta_k(V),\infty) = I_{k,V}(\eta_1(V)-\eta_k(V),1/V^{2\alpha_3}) + I_{k,V}(1/V^{2\alpha_3},\infty). \tag{3.36}$$

Since $\lim_{V\to\infty}\eta_k(V)=0$, by Lemma 3.1 we have

$$\lim_{V\to\infty} F_{k,V}(\eta) = \lim_{V\to\infty} F_V(\eta+\eta_k(V)) = F(\eta) \tag{3.37}$$

and

$$F_{k,V}(\eta) \le C'\eta^{3/2} \tag{3.38}$$

for $\eta > 1/V^{2\alpha_3}$, using the estimate $x - x^2/2 \le \ln(1+x) \le x$ we get

$$\lim_{V\to\infty} I_{k,V}(1/V^{2\alpha_3},\infty) = \lambda\rho_c. \tag{3.39}$$

Let

$$G_{k,V}(\xi) := V F_{k,V}(\xi/V). \tag{3.40}$$

Then

$$I_{k,V}(\eta_1(V)-\eta_k(V),1/V^{2\alpha_3}) = \int\limits_{\left[V(\eta_1(V)-\eta_k(V)),V^{1-2\alpha_3}\right]} \ln\left|1+\frac{1-e^{-\lambda/V}}{e^{\beta\xi/V}-1}\right| G_{k,V}(d\xi). \tag{3.41}$$

Now let $E_k(V)$ correspond to $\epsilon_{(n,1,1),V}$. Then in the limit, $G_{k,V}$ gives a non-trivial point measure concentrated on the set $\{\beta^{-1}\eta_{m,n},\, m\neq n\}$:

$$\lim_{V\to\infty} G_{k,V}(\xi>0) = \#\left\{m : \beta^{-1}\eta_{m,n} \le \xi,\, m\neq n\right\} . \tag{3.42}$$

Therefore, by (3.41) and (3.42) we get

$$\lim_{V\to\infty}\left\{I_{k,V}(V(\eta_1(V)-\eta_k(V)),1/V^{2\alpha_3})\right\} = \sum_{m\neq n}\ln\left|1+\frac{\lambda}{\eta_{m,n}}\right| . \tag{3.43}$$

Thus for $\lambda > |\eta_{1,n}|$,

$$\lim_{V\to\infty}\int\limits_0^\infty K_{k,V}(dx)e^{-\lambda x} = \exp\{-\lambda\rho_c\}\exp\left\{-\sum_{m\neq n}\ln\left|1+\frac{\lambda}{\eta_{m,n}}\right|\right\}. \tag{3.44}$$

and the lemma follows by inverting the Laplace transform. □

Theorem 3.5 *Let $\rho>\rho_c$ and $\alpha_1 = 1/2$. Let $E_k(V)$ correspond to $\epsilon_{(n,1,1),V}$. Then*

$$\lim_{V\to\infty}\langle\exp\{-\lambda N_k/V\}\rangle_V^c(\rho) = \frac{\sum\limits_{m=1,\,m\neq n}^{\infty} b_{m,n}\dfrac{\eta_{m,n}^2}{\eta_{m,n}-\lambda}\{\exp[\lambda(\rho-\rho_c)]-\exp[-\eta_{m,n}(\rho-\rho_c)]\}}{\sum\limits_{m=1,\,m\neq n}^{\infty} b_{m,n}\eta_{m,n}\{1-\exp[-\eta_{m,n}(\rho-\rho_c)]\}}. \tag{3.45}$$

Proof: The identity (3.28) gives:

$$\lim_{V\to\infty} \langle \exp\{-\lambda N_k/V\}\rangle_V^{\mathcal{C}}(\rho) = 1 - \frac{\lambda e^{-\lambda\rho}\displaystyle\int_0^\rho K_k(x)e^{\lambda x}\,dx}{K_k(\rho)}. \tag{3.46}$$

From the preceding lemma and for $\lambda > 0$

$$\begin{aligned}\frac{\lambda e^{-\lambda\rho}\displaystyle\int_0^\rho K_k(x)e^{\lambda x}\,dx}{K_k(\rho)} &= \frac{\displaystyle\sum_{m\neq n}^{\infty} b_{m,n}\eta_{m,n}\lambda e^{-\lambda\rho}\int_{\rho_c}^{\rho} e^{\lambda x}\left\{1-\exp\left[-\eta_{m,n}(x-\rho_c)\right]\right\}dx}{\displaystyle\sum_{m\neq n} b_{m,n}\eta_{m,n}\left\{1-\exp\left[-\eta_{m,n}(\rho-\rho_c)\right]\right\}} \\ &= -\frac{\displaystyle\sum_{m\neq n} b_{m,n}\frac{\eta_{m,n}}{\eta_{m,n}-\lambda}\left\{\eta_{m,n}\exp\left[-\lambda(\rho-\rho_c)\right]-\lambda\exp\left[-\eta_{m,n}(\rho-\rho_c)\right]-(\eta_{m,n}-\lambda)\right\}}{\displaystyle\sum_{m\neq n} b_{m,n}\eta_{m,n}\left\{1-\exp\left[-\eta_{m,n}(\rho-\rho_c)\right]\right\}}.\end{aligned} \tag{3.47}$$

This gives (3.45). □

We are now in a position to prove that in this case there is BEC of type II for $\rho > \rho_c$.

Theorem 3.6 *For $\rho > \rho_c$ and for $\alpha_1 = 1/2$ all states with $\mathbf{n} = (n,1,1)$ are macroscopically occupied (BEC of type II) while all the other states are not. The occupation density is given by*

$$\lim_{V\to\infty}\left\langle \frac{N_{(n,1,1)}}{V}\right\rangle_V^{\mathcal{C}}(\rho) = \frac{\displaystyle\sum_{m=1,\,\neq n}^{\infty} b_{m,n}\left\{\eta_{m,n}(\rho-\rho_c)-1+\exp\left[-\eta_{m,n}(\rho-\rho_c)\right]\right\}}{\displaystyle\sum_{m=1,\,m\neq n}^{\infty} b_{m,n}\eta_{m,n}\left\{1-\exp\left[-\eta_{m,n}(\rho-\rho_c)\right]\right\}},$$

Proof: As we mentioned above it is sufficient to check that there exists $K < \infty$ such that for all V

$$\left\langle (N_k/V)^2\right\rangle_V^{\mathcal{C}}(\rho) < K. \tag{3.48}$$

Then the theorem follows from the preceding one. We have:

$$\begin{aligned}\left\langle (N_k/V)^2\right\rangle_V^{\mathcal{GC}}(\mu_V(\rho)) &= \int_{[0,\infty)} \left\langle (N_k/V)^2\right\rangle_V^{\mathcal{C}}(x)\,\tilde{\mathbb{K}}_V(\rho;dx) \\ &\geq \int_{[\rho,\infty)} \left\langle (N_k/V)^2\right\rangle_V^{\mathcal{C}}(x)\,\tilde{\mathbb{K}}_V(\rho;dx) \\ &\geq \left\langle (N_k/V)^2\right\rangle_V^{\mathcal{C}}(\rho)\,\tilde{\mathbb{K}}_V(\rho;[\rho,\infty)).\end{aligned} \tag{3.49}$$

This implies the existence of K as above since $\left\langle (N_k/V)^2\right\rangle_V^{\mathcal{GC}}(\mu_V(\rho))$ converges as $V\to\infty$ and $\tilde{\mathbb{K}}_V(\rho;[\rho,\infty))$ converges to a non-zero limit. □

3.3 Case $\alpha_1 < 1/2$.

In [5] the canonical PBG in parallelepipeds

$$\Lambda_V := \left\{x \in \mathbb{R}^3 : 0 \leq x_j \leq a_j V^{1/3}, j = 1,2,3\right\}, \quad a_1 a_2 a_3 = 1 \tag{3.50}$$

were considered. It was proved that for this system there is BEC of *type I*. In particular it was proved that:

Proposition 3.1 ([5],Theorem 1) *For the PBG in parallelepipeds (3.50), the following limits hold when $\lambda \in \mathbb{R}$:*

$$\lim_{V\to\infty} \langle \exp\{-\lambda N_k/V\}\rangle_V^c(\rho) = \begin{cases} \exp\{-\lambda(\rho-\rho_c\}, & \text{for } k = 1, \\ 1, & \text{for } k > 1, \end{cases} \tag{3.51}$$

if $\rho > \rho_c$, and

$$\lim_{V\to\infty} \langle \exp\{-\lambda V^\gamma N_k/V\}\rangle_V^c(\rho) = 1 \tag{3.52}$$

for any $0 \leq \gamma < 1$ and $k \geq 1$, if $\rho \leq \rho_c$.

Note that in [5], Theorem 1, $\lambda \leq 0$ but this is not necessary.

In this section we shall prove a similar result to Proposition 3.1 for case of the rectangular parallelepipeds (1.1) with $\alpha_1 < 1/2$, that is, we shall show that in this case there is also *type I* BEC. It is sufficient to show that there is condensation in the ground state since by Theorem 3.3 no other state can be macroscopically occupied.

Let $K_{1,V}(dx)$ be as in (3.27) in Section 3.2.

Lemma 3.4 *Let $\alpha_1 < 1/2$. Then*

$$\lim_{V\to\infty} K_{1,V}(dx) = \delta_{\rho_c}(dx). \tag{3.53}$$

Proof: The proof is almost identical to that of Lemma 3.3. The only difference is that since $\eta_j(V) - \eta_1(V) \geq a_{j,k}/V^{2\alpha_1}$, (3.40) implies that

$$G_{1,V}(\xi > 0) = \#\{j : \eta_j(V) - \eta_1(V) \leq \xi/V\} \to 0 \text{ when } V \to \infty, \tag{3.54}$$

for $\alpha_1 < 1/2$ and the lemma follows. $\square$

We can now prove that in this case there is BEC of type I for $\rho > \rho_c$.

Theorem 3.7 *For $\rho > \rho_c$ and for $\alpha_1 < 1/2$ only the ground-state is macroscopically occupied (BEC of type I):*

$$\lim_{V\to\infty} \frac{1}{V} \langle N_{\mathbf{n}}\rangle_V^c(\rho) = \begin{cases} \rho - \rho_c, & \text{for } \mathbf{n} = (1,1,1), \\ 0, & \text{for } \mathbf{n} \neq (1,1,1). \end{cases} \tag{3.55}$$

Proof: From the preceding lemma and the identity (3.28) we have for $\lambda > 0$

$$\lim_{V\to\infty} \langle \exp\left(-\lambda N_{\mathbf{n}}/V\right)\rangle_V^{\mathcal{GC}}(\rho) = -\lambda(\rho - \rho_c). \tag{3.56}$$

It is sufficient to show the second moment is bounded, that is, there exists $K < \infty$ such that for all V

$$\left\langle (N_k/V)^2 \right\rangle_V^{\mathcal{C}}(\rho) < K. \tag{3.57}$$

The bound can be obtained by the same argument as in Theorem 3.6. □

In the rest of this subsection we study the fluctuations of N_1/V. We need the shifted integrated density of states in d dimensions $F_1^{(d)}$, $d = 1, 2, 3$, in the unit boxes $[0, 1]^d$:

$$F_1^{(d)}(\eta) := \#\left\{\mathbf{n} \mid \mathbf{n} \in \mathbb{N}^d,\ \ \frac{\pi^2}{2}\sum_{j=1}^{d}(n_j - 1)^2 \le \eta\right\}, \tag{3.58}$$

Theorem 3.8 *Suppose $\alpha_1 < 1/2$ and let $\gamma = 1 - 2\alpha_1 > 0$. Then for $\rho > \rho_c$,*

$$\lim_{V\to\infty} \langle \exp\left\{\lambda V^\gamma (N_1/V - \langle N_1/V\rangle_V^{\mathcal{C}}(\rho))\right\}\rangle_V^{\mathcal{C}}(\rho) =$$

$$\begin{cases} \exp\left(g_1(\lambda)\right), & \text{if } \alpha_3 < \alpha_2 < \alpha_1 < 1/2, \\ \exp\left(2g_1(\lambda) + g_2(\lambda)\right), & \text{if } \alpha_3 < \alpha_2 = \alpha_1 < 1/2, \\ \exp\left(3g_1(\lambda) + 3g_2(\lambda) + g_3(\lambda)\right), & \text{if } \alpha_3 = \alpha_2 = \alpha_1 = 1/3, \end{cases} \tag{3.59}$$

where

$$\begin{aligned} g_1(\lambda) &= \int_{(0,\infty)} \left[-\ln\left(1 + \frac{\lambda}{\beta\eta}\right) + \frac{\lambda}{\beta\eta}\right] F_1^{(1)}(d\eta), \\ g_2(\lambda) &= \int_{(0,\infty)^2} \left[-\ln\left(1 + \frac{\lambda}{\beta(\eta_1+\eta_2)}\right) + \frac{\lambda}{\beta(\eta_1+\eta_2)}\right] F_1^{(2)}(d\eta_1, d\eta_2), \\ g_3(\lambda) &= \int_{(0,\infty)^3} \left[-\ln\left(1 + \frac{\lambda}{\beta(\eta_1+\eta_2+\eta_3)}\right) + \frac{\lambda}{\beta(\eta_1+\eta_2+\eta_3)}\right] F_1^{(3)}(d\eta_1, d\eta_2, d\eta_3). \end{aligned} \tag{3.60}$$

Remark: Note that $3g_1(\lambda) + 3g_2(\lambda) + g_3(\lambda)$ is the same as $g(\lambda)$ in ([5]).
Proof: Let

$$L_V(x) :=$$
$$\begin{cases} Z_V(r)\exp\left\{-\beta(Vp_1 - rE_k(V)\right\}, & \text{for } V^\gamma(r/V - \rho_c^V)) < x \le V^\gamma((r+1)/V - \rho_c^V), \\ 0, & \text{for } x \le -V^\gamma \rho_c^V, \end{cases}$$

for $r = 0, 1, 2, \ldots$, where p_1 is as in (3.30),

$$\rho_c^V := \int_{(0,\infty)} \frac{1}{e^{\beta\eta} - 1} F_V(d\eta) \tag{3.61}$$

and where we put $Z_V(0) = 1$. Then

$$\langle \exp\{\lambda V^{\gamma}(N_1/V - (\rho - \rho_c^{V}))\}\rangle_V^{\mathcal{C}}(\rho) = \int\limits_{(-\infty,\,\alpha_V)} e^{-\lambda x} L_V(dx)/K_{1,V}(\rho + 1/V) \qquad (3.62)$$

where $\alpha_V = V^{\gamma}(\rho - \rho_c^{V}) + V^{-2\alpha_1}$. By Lemma 3.4 for $\rho > \rho_c$, $\lim_{V\to\infty} K_{1,V}(\rho + 1/V) = 1$ and therefore

$$\langle \exp\{\lambda V^{\gamma}(N_1/V - (\rho - \rho_c^{V}))\}\rangle_V^{\mathcal{C}}(\rho) = \lim_{V\to\infty} \int\limits_{(-\infty,\,\infty)} e^{-\lambda x} L_V(dx). \qquad (3.63)$$

Now

$$\begin{aligned} \ln \int\limits_{(-\infty,\,\infty)} e^{-\lambda x} L_V(dx) &= V \int\limits_{(0,\,\infty)} \left[-\ln\left(1 + \frac{1 - e^{-\lambda V^{-2\alpha_1}}}{e^{\beta\eta} - 1}\right) + \frac{\lambda V^{-2\alpha_1}}{e^{\beta\eta} - 1} \right] F_V(d\eta) \\ &= \sum_{\mathbf{n} \neq (1,1,1)} \left[-\ln\left(1 + \frac{1 - e^{-\lambda V^{-2\alpha_1}}}{e^{\beta(\epsilon_{\mathbf{n},V} - \epsilon_{(1,1,1),V})} - 1}\right) + \frac{\lambda V^{-2\alpha_1}}{e^{\beta(\epsilon_{\mathbf{n},V} - \epsilon_{(1,1,1),V})} - 1} \right]. \end{aligned} \qquad (3.64)$$

Consider first the case $\alpha_3 < \alpha_2 < \alpha_1 < 1/2$. We write

$$\ln \int\limits_{(-\infty,\,\infty)} e^{-\lambda x} L_V(dx) = A_V + B_V \qquad (3.65)$$

where

$$A_V = \sum_{n_1 \neq 1} \left[-\ln\left(1 + \frac{1 - e^{-\lambda V^{-2\alpha_1}}}{e^{\beta(\epsilon_{(n_1,1,1),V} - \epsilon_{(1,1,1),V})} - 1}\right) + \frac{\lambda V^{-2\alpha_1}}{e^{\beta(\epsilon_{(n_1,1,1),V} - \epsilon_{(1,1,1),V})} - 1} \right] \qquad (3.66)$$

and

$$B_V = \sum_{(n_2,n_3) \neq (1,1)} \left[-\ln\left(1 + \frac{1 - e^{-\lambda V^{-2\alpha_1}}}{e^{\beta(\epsilon_{\mathbf{n},V} - \epsilon_{(1,1,1),V})} - 1}\right) + \frac{\lambda V^{-2\alpha_1}}{e^{\beta(\epsilon_{\mathbf{n},V} - \epsilon_{(1,1,1),V})} - 1} \right]. \qquad (3.67)$$

Note that by definition (3.58)

$$A_V = \int\limits_{(0,\,\infty)} \left[-\ln\left(1 + \frac{1 - e^{-\lambda V^{-2\alpha_1}}}{e^{\beta V^{-2\alpha_1}\eta} - 1}\right) + \frac{\lambda V^{-2\alpha_1}}{e^{\beta V^{-2\alpha_1}\eta} - 1} \right] F_1^{(1)}(d\eta). \qquad (3.68)$$

Using the bounds (obtained from the inequality $-x < -\ln(1+x) < \frac{1}{2}x^2 - x$):

$$\begin{aligned} 0 \; &< \; \frac{\lambda - 1 + e^{-\lambda}}{e^{x} - 1} < -\ln\left(1 + \frac{1 - e^{-\lambda}}{e^{x} - 1}\right) + \frac{\lambda}{e^{x} - 1} \\ &< \; \frac{1}{2}\left(\frac{1 - e^{-\lambda}}{e^{x} - 1}\right)^2 + \frac{\lambda - 1 + e^{-\lambda}}{e^{x} - 1} < \frac{c\lambda^2}{x^2}, \end{aligned} \qquad (3.69)$$

we get using the Dominated Convergence Theorem

$$\lim_{V\to\infty} A_V = g_1(\lambda). \tag{3.70}$$

Using the same inequality

$$\begin{aligned}
0 < B_V &\le \frac{\lambda^2}{\beta^2 V^{4\alpha_1}} \sum_{(n_2,n_3)\neq(1,1)} \frac{1}{(\epsilon_{\mathbf{n},V} - \epsilon_{(1,1,1),V})^2} \\
&= \frac{4\lambda^2}{\pi^4\beta^2 V^{4\alpha_1}} \sum_{(n_2,n_3)\neq(1,1)} \frac{1}{\left(\frac{(n_1^2-1)}{V^{2\alpha_1}} + \frac{(n_2^2-1)}{V^{2\alpha_2}} + \frac{(n_3^2-1)}{V^{2\alpha_3}}\right)^2} \\
&= \frac{4\lambda^2}{\pi^4\beta^2} \sum_{(n_2,n_3)\neq(1,1)} \frac{1}{\left((n_1^2-1) + (n_2^2-1)V^{2(\alpha_1-\alpha_2)} + (n_3^2-1)V^{2(\alpha_1-\alpha_2)}\right)^2}.
\end{aligned} \tag{3.71}$$

Now the summand in the last sum tends to zero as $V \to \infty$ and it is bounded above by

$$\frac{1}{\left((n_1^2-1) + (n_2^2-1) + (n_3^2-1)\right)^2}. \tag{3.72}$$

Since

$$\sum_{(n_2,n_3)\neq(1,1)} \frac{1}{\left((n_1^2-1) + (n_2^2-1) + (n_3^2-1)\right)^2} < \infty, \tag{3.73}$$

$B_V \to 0$ as $V \to \infty$ by the same theorem.

Next we consider the case $\alpha_3 < \alpha_2 = \alpha_1 < 1/2$. Now we take

$$A_V = \sum_{(n_1,n_2)\neq(1,1)} \left[-\ln\left(1 + \frac{1 - e^{-\lambda V^{-2\alpha_1}}}{e^{\beta(\epsilon_{(n_1,n_2,1),V} - \epsilon_{(1,1,1),V})} - 1}\right) + \frac{\lambda V^{-2\alpha_1}}{e^{\beta(\epsilon_{(n_1,n_2,1),V} - \epsilon_{(1,1,1),V})} - 1}\right] \tag{3.74}$$

and

$$B_V = \sum_{n_3\neq 1} \left[-\ln\left(1 + \frac{1 - e^{-\lambda V^{-2\alpha_1}}}{e^{\beta(\epsilon_{\mathbf{n},V} - \epsilon_{(1,1,1),V})} - 1}\right) + \frac{\lambda V^{-2\alpha_1}}{e^{\beta(\epsilon_{\mathbf{n},V} - \epsilon_{(1,1,1),V})} - 1}\right]. \tag{3.75}$$

In this case by definition (3.58)

$$\begin{aligned}
A_V &= 2\sum_{n_1\neq 1} \left[-\ln\left(1 + \frac{1 - e^{-\lambda V^{-2\alpha_1}}}{e^{\beta(\epsilon_{(n_1,1,1),V} - \epsilon_{(1,1,1),V})} - 1}\right) + \frac{\lambda V^{-2\alpha_1}}{e^{\beta(\epsilon_{(n_1,1,1),V} - \epsilon_{(1,1,1),V})} - 1}\right] \\
&\quad + \sum_{n_1\neq 1, n_2\neq 1} \left[-\ln\left(1 + \frac{1 - e^{-\lambda V^{-2\alpha_1}}}{e^{\beta(\epsilon_{(n_1,n_2,1),V} - \epsilon_{(1,1,1),V})} - 1}\right) + \frac{\lambda V^{-2\alpha_1}}{e^{\beta(\epsilon_{(n_1,n_2,1),V} - \epsilon_{(1,1,1),V})} - 1}\right] \\
&= 2\int_{(0,\infty)} \left[-\ln\left(1 + \frac{1 - e^{-\lambda V^{-2\alpha_1}}}{e^{\beta V^{-2\alpha_1}\eta} - 1}\right) + \frac{\lambda V^{-2\alpha_1}}{e^{\beta V^{-2\alpha_1}\eta} - 1}\right] F_1^{(1)}(d\eta) \\
&\quad + \int_{(0,\infty)^2} \left[-\ln\left(1 + \frac{1 - e^{-\lambda V^{-2\alpha_1}}}{e^{\beta V^{-2\alpha_1}(\eta_1+\eta_2)} - 1}\right) + \frac{\lambda V^{-2\alpha_1}}{e^{\beta V^{-2\alpha_1}(\eta_1+\eta_2)} - 1}\right] F_1^{(2)}(d\eta_1, d\eta_2).
\end{aligned} \tag{3.76}$$

By the same argument as above

$$\lim_{V\to\infty} A_{V} = 2g_1(\lambda) + g_2(\lambda) \tag{3.77}$$

and

$$\lim_{V\to\infty} B_{V} = 0. \tag{3.78}$$

Finally for $\alpha_3 = \alpha_2 = \alpha_1 = 1/3$

$$\begin{aligned}
&\int\limits_{(-\infty,\infty)} L_{V}(dx)e^{-\lambda x} \\
&= 3\sum_{n_1\neq 1}\left[-\ln\left(1+\frac{1-e^{-\lambda V^{-2\alpha_1}}}{e^{\beta(\epsilon_{(n_1,1,1),V}-\epsilon_{(1,1,1),V})}-1}\right)+\frac{\lambda V^{-2\alpha_1}}{e^{\beta(\epsilon_{(n_1,1,1),V}-\epsilon_{(1,1,1),V})}-1}\right] \\
&+3\sum_{n_1\neq 1,n_2\neq 1}\left[-\ln\left(1+\frac{1-e^{-\lambda V^{-2\alpha_1}}}{e^{\beta(\epsilon_{(n_1,n_2,1),V}-\epsilon_{(1,1,1),V})}-1}\right)+\frac{\lambda V^{-2\alpha_1}}{e^{\beta(\epsilon_{(n_1,n_2,1),V}-\epsilon_{(1,1,1),V})}-1}\right] \\
&= 3\int\limits_{(0,\infty)}\left[-\ln\left(1+\frac{1-e^{-\lambda V^{-2\alpha_1}}}{e^{\beta V^{-2\alpha_1}\eta}-1}\right)+\frac{\lambda V^{-2\alpha_1}}{e^{\beta V^{-2\alpha_1}\eta}-1}\right]F_1^{(1)}(d\eta) \\
&+3\int\limits_{(0,\infty)^2}\left[-\ln\left(1+\frac{1-e^{-\lambda V^{-2\alpha_1}}}{e^{\beta V^{-2\alpha_1}(\eta_1+\eta_2)}-1}\right)+\frac{\lambda V^{-2\alpha_1}}{e^{\beta V^{-2\alpha_1}(\eta_1+\eta_2)}-1}\right]F_1^{(2)}(d\eta_1,d\eta_2) \\
&+\int\limits_{(0,\infty)^3}\left[-\ln\left(1+\frac{1-e^{-\lambda V^{-2\alpha_1}}}{e^{\beta V^{-2\alpha_1}(\eta_1+\eta_2+\eta_3)}-1}\right)+\frac{\lambda V^{-2\alpha_1}}{e^{\beta V^{-2\alpha_1}(\eta_1+\eta_2+\eta_3)}-1}\right]F_1^{(3)}(d\eta_1,d\eta_2,d\eta_3).
\end{aligned} \tag{3.79}$$

Thus

$$\lim_{V\to\infty}\ln\int\limits_{(-\infty,\infty)} e^{-\lambda x}L_{V}(dx) = 3g_1(\lambda)+3g_2(\lambda)+g_3(\lambda). \tag{3.80}$$

We have therefore proved that

$$\begin{aligned}
&\lim_{V\to\infty}\langle\exp\{\lambda V^{\gamma}(N_1/V-(\rho-\rho_c^{V}))\}\rangle_{V}^{c}(\rho) = \\
&\qquad\begin{cases}\exp\left(g_1(\lambda)\right), & \text{if } \alpha_3<\alpha_2<\alpha_1<1/2,\\ \exp\left(2g_1(\lambda)+g_2(\lambda)\right), & \text{if } \alpha_3<\alpha_2=\alpha_1<1/2,\\ \exp\left(3g_1(\lambda)+3g_2(\lambda)+g_3(\lambda)\right), & \text{if } \alpha_3=\alpha_2=\alpha_1=1/3.\end{cases}
\end{aligned}$$

To finish the proof we centre the distribution about $\langle N_1/V\rangle_{V}^{c}(\rho)$. From (3.62) we get

$$\begin{aligned}
\left\langle\{\lambda V^{\gamma}(N_1/V-(\rho-\rho_c^{V}))\}^2\right\rangle_{V}^{c}(\rho) &= \int\limits_{(-\infty,\alpha_V)} x^2L_{V}(dx)/K_{1,V}(\rho+1/V) \\
&\leq 2\int\limits_{(-\infty,\infty)} x^2L_{V}(dx).
\end{aligned}$$

Using (3.64) this gives

$$\begin{aligned}\left\langle [V^{\gamma}(N_1/V-(\rho-\rho_c^{V}))]^2\right\rangle_V^{\mathcal{C}}(\rho) &\le V^{1-4\alpha_1}\int_{(-\infty,\infty)}\left\{\frac{1}{2(e^{\beta\eta}-1)}+\frac{1}{(e^{\beta\eta}-1)^2}\right\}F_V(d\eta)\\ &\le \frac{3V^{1-4\alpha_1}}{2}\int_{(-\infty,\infty)}\frac{1}{\eta^2}F_V(d\eta). \qquad (3.81)\end{aligned}$$

Thus by the same arguments as above this second moment is bounded. Since $g_1'(0)=g_2'(0)=g_3'(0)$ we then have

$$\lim_{V\to\infty}V^{\gamma}(\langle N_1/V\rangle_V^{\mathcal{C}}(\rho)-(\rho-\rho_c^{V}))=\lim_{V\to\infty}\langle V^{\gamma}(N_1/V-(\rho-\rho_c^{V}))\rangle_V^{\mathcal{C}}(\rho)=0 \qquad (3.82)$$

completing the proof. □

4 Conclusion

(a) Since the paper by Buffet and Pulé [5] it has been known that there are differences in the fluctuations of the PBG condensate in the canonical and grand-canonical ensembles. We have shown that the picture becomes even more complicated if one passes to the case of Casimir boxes where there is already generalized BEC in the GCE.

We have shown in general that there is a kind of *stability principle* relating the two ensembles: condensation in the GCE is more stable than in the CE. In fact we proved (Theorem 3.3) that that the absence of the macroscopically occupied single-particle states in GCE implies the same in the CE, whereas the converse is not necessarily true. However in the case of the Casimir boxes considered here the two ensembles exhibit the same types of BEC for the same geometry. What varies in some cases is the fluctuations and the amount of condensate in the individual levels.

(b) As we have mentioned in Section 1, BEC of *type* I and II is also known in the literature describing the experiments with trapped bosons as "fragmentation" of the condensate, [9], [16], [8]. In these papers the authors relate this phenomenon to the interaction properties of the Bose gas, arguing that it is the exchange interaction that causes bosons with repulsive interaction to condense into a single one-particle state whereas for attractive interactions the condensate may be "fragmented" into a number of degenerate or nearly degenerate single particle states, see [14],[17]. Here we have shown that BEC of type II occurs in the *non-interacting* Bose gas and that this is due simply to a geometric anisotropy of the boxes known since Casimir [7]. On the other hand, there are exactly soluble models (in cubic boxes) showing that some truncated repulsive interactions are able to convert BEC in the ground state into a generalized condensation of type III (see [11], [4]). In [15] an even simpler repulsive interaction than in [11] and [4] is proposed that produces type I condensation in a few degenerate single-particle states.

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